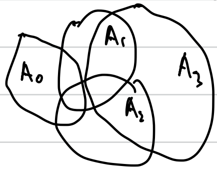
Measure Theory with Ergodic Horizons
Lecture 4

Before proving Claim (c), let's record a couple of easy tools for working with algebras and finitely additive measures.

Disjointification trick. For an algebra $\mathcal{A}$, any ctbl union $\bigcup_n A_n$ of sets in $\mathcal{A}$ is equal to a ctbl disjoint union $\bigsqcup_n \tilde{A}_n$ of sets in $\mathcal{A}$ with $\tilde{A}_n \subseteq A_n$. In fact $\bigcup_{i<n} A_i = \bigcup_{i<n} \tilde{A}_i$ for all n.

Proof.



Let $\tilde{A}_n := A_n \setminus \bigcup_{i<n} A_i$. $\square$

Properties of finitely additive measures. Let μ be a finitely additive measure on an algebra $\mathcal{A}$ on a set X. Then:

(a) μ is monotone: if $A \subseteq B$ are in $\mathcal{A}$, then $\mu(A) \leq \mu(B)$.

(b) μ is ctbly superadditive: $\mu\left(\bigsqcup_{n\in\mathbb{N}} A_n\right) \geq \sum_{n\in\mathbb{N}} \mu(A_n)$ for disjoint $A_n \in \mathcal{A}$ with $\bigsqcup_n A_n \in \mathcal{A}$.

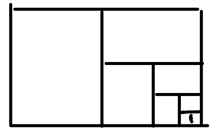
(c) μ is finitely subadditive*: $\mu\left(\bigcup_{n<N} A_n\right) \leq \sum_{n<N} \mu(A_n)$ for $A_n \in \mathcal{A}$.

(c') If μ is ctbly additive, then it is ctbly subadditive*: $\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) \leq \sum_{n\in\mathbb{N}} \mu(A_n)$ for $A_n \in \mathcal{A}$ with $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$.

Proof. (a) By finite additivity, $\mu(B) = \mu(A) + \mu(B\setminus A) \geq \mu(A)$.

(b) $\forall N \in \mathbb{N}$, by fin. add., $\sum_{n<N} \mu(A_n) = \mu\left(\bigsqcup_{n<N} A_n\right) \leq \mu\left(\bigsqcup_{n\in\mathbb{N}} A_n\right)$ by monotonicity, but N is arbitrary, so $\sum_{n\in\mathbb{N}} \mu(A_n) \leq \mu\left(\bigsqcup_{n\in\mathbb{N}} A_n\right)$.

(c)-(c') By disjointification and monotonicity. $\square$



Claim (c). λ is ctbly additive.

Proof. We only the special case when a bdd box B is partitioned into infinitely many boxes: $B = \bigsqcup_{n\in\mathbb{N}} B_n$. The general case follows from this special case and is left as HW.

So, we suppose that B is bdd. In the case of Bernoulli, we used that B was compact and the B_n were open, while in our case neither may be true. However, we can replace B with a closed box $B' \subseteq B$ with $\lambda(B \setminus B') < \varepsilon/2$ for some apriori fixed arbitrary $\varepsilon > 0$. Similarly, we can replace B_n with an open box $\tilde{B}_n \supseteq B_n$ with $\lambda(\tilde{B}_n \setminus B_n) < \frac{\varepsilon}{2\cdot 2^{n+1}}$. Thus $\{\tilde{B}_n\}_{n\in\mathbb{N}}$ is an open cover of the compact set B', hence there is a finite subcover $\tilde{B}_0, \tilde{B}_1, \dots, \tilde{B}_N$.

Notation. For reals $a, b \in \mathbb{R}$, we write $a \approx_\varepsilon b$ if $|a-b| \le \varepsilon$. We also write $a \le_\varepsilon b$ (resp. $a \ge_\varepsilon b$) if $a \le b + \varepsilon$ (resp. $a \ge b - \varepsilon$).

Now
$$\lambda(B) \approx_{\varepsilon/2} \lambda(B') \overset{(a)}{\le} \lambda\Big(\bigcup_{n\le N} \tilde{B}_n\Big) \overset{(c)}{\le} \sum_{n\le N} \lambda(\tilde{B}_n) \le \sum_{n\in\mathbb{N}} \lambda(\tilde{B}_n) \approx_{\varepsilon/2} \sum_{n\in\mathbb{N}} \lambda(B_n),$$
so λ is ctbly subadditive, hence ctbly additive by (b) above. $\square$

Carathéodory extension.

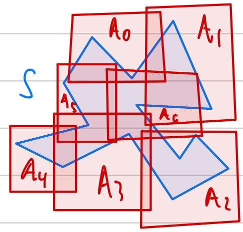
Having defined a premeasure on an algebra $\mathcal{A}$, we would like to extend it to the σ-algebra $\langle \mathcal{A} \rangle_\sigma$ generated by $\mathcal{A}$, in other words, on any σ-algebra $\mathcal{M} \supseteq \mathcal{A}$.

Carathéodory's extension theorem. Every premeasure μ on an algebra $\mathcal{A}$ on X extends to a measure $\tilde{\mu}$ on the σ-algebra $\langle \mathcal{A} \rangle_\sigma$. If μ is σ-finite ($X = \bigsqcup_{n\in\mathbb{N}} X_n$ with $X_n \in \mathcal{A}$ and $\mu(X_n) < \infty$), then the extension is unique.

To prove this, we need the following concept.

Def. For a finitely additive measure μ on an algebra $\mathcal{A}$ on X, we define its *outer measure* $\mu^*: \mathcal{P}(X) \to [0,\infty]$ by setting, for each $S \subseteq X$,

$$\mu^*(S) := \inf \left\{ \sum_{n\in\mathbb{N}} \mu(A_n) : \bigcup_{n\in\mathbb{N}} A_n \supseteq S \text{ and } A_n \in \mathcal{A} \right\}.$$



Properties of outer measures. Let μ be a premeasure on an algebra $\mathcal{A}$. Then its outer measure is:

(a) monotone: if $A \subseteq B$ then $\mu^*(A) \leq \mu^*(B)$.

(b) ctbly-subadditive*: $\mu^*\left(\bigcup_{n\in\mathbb{N}} S_n\right) \leq \sum_{n\in\mathbb{N}} \mu^*(S_n)$ for arbitrary sets $S_n \subseteq X$.

Proof. HW.

Prop. For a premeasure μ on an algebra $\mathcal{A}$, its outer measure $\mu^*|_{\mathcal{A}} = \mu$.

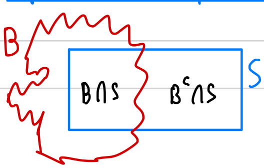
Proof. Let $A \in \mathcal{A}$. By def, $\mu^*(A) \leq \mu(A)$, so we only need to show that $\mu(A) \leq \mu^*(A)$. Let $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{A}$ be a cover of A. By replacing A_n with $A_n \cap A$ and disjointifying, we may assume that $A = \bigsqcup_{n\in\mathbb{N}} A_n$ hence $\mu(A) = \sum_{n\in\mathbb{N}} \mu(A_n)$ by ctbl additivity of μ. $\square$

Carathéodory's extension (existence). Every premeasure μ on an algebra $\mathcal{A}$ on X extends to a measure $\tilde{\mu}$ on the σ-algebra $\langle \mathcal{A} \rangle_\sigma$.

Proof. It is enough to show that the outer measure μ^* is finitely additive on $\langle \mathcal{A} \rangle_\sigma$ because it would be automatically ctbly super and sub additive.

Top-to-bottom proof (by Carathéodory). Say that a set $B \subseteq X$ *butchers* a set $S \subseteq X$ if

$$\mu^*(S) < \mu^*(B \cap S) + \mu^*(B^c \cap S).$$



Say that B is *conservative* if it doesn't butcher any set.

Let $\mathcal{M}$ be the collection of all conservative sets. We then show that
(i) $\mathcal{M} \supseteq \mathcal{A}$.
(ii) $\mathcal{M}$ is an σ-algebra.
It then follows from the definition of conservative sets that μ^* is finitely additive on $\mathcal{M}$, finishing the proof. The proofs of (i) and (ii) are outlined in HW. $\square$

Bottom-up proof (by Tao). This proof only works for σ-finite premeasures. Firstly, we assume that μ is finite, i.e. $\mu(X) < \infty$. The general σ-finite case follows from this special case by considering a partition $X = \bigsqcup_{n\in\mathbb{N}} X_n$ where $X_n \in \mathcal{A}$ and $\mu(X_n) < \infty$ and putting together the extensions we obtain for each X_n. Left for HW.

We now show that the function $d : \mathcal{P}(X) \times \mathcal{P}(X) \to [0, \mu(X)]$ defines a pseudo-metric on $\mathcal{P}(X)$,
$$(A, B) \mapsto \mu^*(A \triangle B)$$
i.e. it's a metric but the axiom "$d(A,B) = 0 \Rightarrow A = B$" may not hold.

The secret of symmetric differences $\triangle$: $A \triangle B = (A \setminus B) \cup (B \setminus A)$. On the level of indicator functions $\mathbb{1}_A, \mathbb{1}_B \in 2^X = \{0,1\}^X$, this is $\mathbb{1}_{A\triangle B} = \mathbb{1}_A \text{ xor } \mathbb{1}_B$ or $\mathbb{1}_A +_2 \mathbb{1}_B$. This is to say that $(\mathcal{P}(X), \triangle)$ forms an abelian group, where $\emptyset$ is the identity and every element is order 2: $A \triangle A = \emptyset$.

Claim (a). d is a pseudo-metric on $\mathcal{P}(X)$.
Proof. Symmetry of d is obvious, so we only verify the triangle inequality. Note that for all $A, B, C \subseteq X$,
$$A \triangle C \underset{\uparrow\ \triangle \text{ is a group operation}}{=} (A \triangle B) \triangle (B \triangle C) \subseteq (A \triangle B) \cup (B \triangle C),$$
so
$$\mu^*(A \triangle C) \underset{\text{monotonicity}}{\leq} \mu^*((A\triangle B) \cup (B \triangle C)) \underset{\text{subadditivity}^*}{\leq} \mu^*(A \triangle B) + \mu^*(B \triangle C) = d(A,B) + d(B,C).$$ $\square$